Overview

First order systems are, by definition, systems whose input-output relationship is a first order differential equation. A first order differential equation contains a first order derivative but no derivative higher than first order – the order of a differential equation is the order of the highest order derivative present in the equation.

First order systems contain a single energy storage element. In general, the order of the input-output differential equation will be the same as the number of independent energy storage elements in the system. Independent energy storage cannot be combined with other energy storage elements to form a single equivalent energy storage element. For example, we previously learned that two capacitors in parallel can be modeled as a single equivalent capacitor – therefore, a parallel combination of two capacitors forms a single independent energy storage element.

First order systems are an extremely important class of systems. Many practical systems are first order; for example, the mass-damper system and the mass heating examples from chapter 2.0 are both first order systems. Higher order systems can often be approximated as first order systems to a reasonable degree of accuracy if they have a dominant first order mode. (System modes will most likely be discussed in later classes in the engineering curriculum.) Understanding first order systems and their responses is an important aspect to design and analysis of systems in general.

In this chapter, we introduce some basic nomenclature relative to first order system responses and illustrate these terms in the context of an example for which the reader may have an intuitive understanding: a mass sliding on a surface. This example, though not directly relevant to the study of electrical circuits, is intended to allow the reader to develop some physical insight into the terminology and concepts relative to the solution of first order differential equations. The concepts and results obtained with this example are then generalized to apply to any arbitrary first order system. These results are used in later chapters to provide insight in the analysis of electrical circuits, for which the student may not yet have an intuitive understanding.
Before beginning this chapter, you should be able to:

- Integrate functions of one variable
- Represent systems in block diagram form (Chapter 1.7.0)
- Write the governing equation for a mass-damper system (Chapter 1.7.0)
- Define, from memory, the time constant of an exponential function (Chapter 2.1)
- Sketch a decaying exponential function, given the function’s initial value and time constant (Chapter 2.1)
- Sketch the unit step function (Chapter 2.1)

After completing this chapter, you should be able to:

- Write the general form of the differential equation governing a first order system
- State, in physical terms, the significance of a differential equation’s homogeneous and particular solutions
- Define, from memory, the relationships between a system’s unforced response, zero-input response, natural response, and the homogeneous solution to the differential equation governing the system
- Define, from memory, the relationships between a system’s forced response, zero-state response, and the particular solution to the differential equation governing the system
- Determine the time constant of a first order system from the differential equation governing the system
- Write mathematical expressions from memory, giving the form of the natural and step responses of a first order system
- Sketch the natural response of a first order system from the differential equation governing the system and the system’s initial condition
- Sketch the step response of a first order system from the differential equation governing the system and the amplitude of the input step function

This chapter requires:

- N/A
Before discussing first-order electrical systems specifically, we will introduce the response of general first order systems. A general first order system is governed by a differential equation of the form:

$$ a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t), \quad t > t_0 $$

(1)

where \( f(t) \) is the (known) input to the system and \( y(t) \) is the response of the system. \( a_1 \) and \( a_0 \) are constants specific to the system being analyzed. We assume in equation (1) that the input function is applied only for times \( t > t_0 \). Thus, from equation (1), we can only determine the response of the system for times \( t > t_0 \).

In order to find the solution to equation (1), we require knowledge of the system’s initial condition:

$$ y(t = t_0) = y_0 $$

(2)

The initial condition, \( y_0 \), defines the state of the system at time \( t = t_0 \). Since equation (1) describes a system which stores energy, the effect of the initial condition is to provide information as to the amount of energy stored in the system at time \( t = t_0 \).

The system described by equations (1) and (2) can be illustrated in block diagram form as shown in Figure 1. The output of the system depends upon the initial condition, \( y_0 \), and the input function \( f(t) \). The initial condition provides information relative to the energy stored in the system prior to application of the input function. The input function provides information relative to the energy being applied to the system from external sources. The input-output equation describes how the system transfers the energy initially present in the system and the energy added to the system to the system output.

![System block diagram](image)

Figure 1. System block diagram.

The solution to equation (1) consists of two parts – the homogeneous solution, \( y_h(t) \), and the particular solution, \( y_p(t) \), as shown below:

$$ y(t) = y_h(t) + y_p(t) $$

(3)

The homogeneous solution is due to the properties of the system and the initial conditions applied to the system; it describes the response of the system if no input is applied to the system, so \( f(t) = 0 \). The homogeneous solution is sometimes called the system’s natural response, the unforced response, or the zero input response. Since all physical systems dissipate energy (according to the second law of Thermodynamics) the homogeneous solution must die out with time; thus, \( y_p(t) \to 0 \) as \( t \to \infty \).
The particular solution describes the system's response to the particular forcing function applied to the system; the form of the particular solution is dictated by the form of the forcing function applied to the system. The particular solution is also called the forced response or the zero state response.

Since we are concerned only with linear systems, superposition principles are applicable, and the overall system response is the sum of the homogeneous and particular solutions. Thus, equation (3) provides the system's overall response to both initial conditions and the particular forcing function being applied to the system.

The previous concepts are rather abstract, so we provide below an example of the application of the above concepts to a system for whose response the students should have some intuitive expectations. This example is intended to provide some physical insight into the concepts presented above prior to applying these concepts to electrical systems.

**Example: Mass-damper system**

As an example of a system which includes energy storage elements we revisit the mass-damper system of chapter 2.0. The system under consideration is shown in Figure 1. The applied force $F(t)$ pushes the mass to the right. The mass's velocity is $v(t)$. The mass slides on a surface with sliding coefficient of friction $b$, which induces a force which opposes the mass's motion. The mass will have some initial velocity:

$$v(t = 0) = v_0$$

(4)

Consistent with chapter 2.0, we consider the applied force to be the input to our system and the mass's velocity to be the output. Figure 2 illustrates the system input-output relationship and initial conditions in block diagram form.

The governing equation for the system was determined in chapter 1.7.0 to be the first order differential equation:

$$m \frac{dv(t)}{dt} + bv(t) = F(t)$$

(5)
We consider two cases of specific forcing functions in the following cases. In the first case, the forcing function is zero, and we determine the system’s natural response or the homogeneous solution to equation (5) above. In the second case, the forcing function is a constant nonzero force applied to the mass with zero initial initial velocity.

**Case i: Natural (homogeneous) response**

Let us consider first the case in which the mass has some initial velocity but no external force is applied to the mass. Intuitively, we expect that the velocity of the mass will decrease until the mass comes to rest. In this example, we will determine the solution of the differential equation (5) and compare this solution with our expectations.

With no applied forcing function, the differential equation governing the system are:

\[ m \frac{dv(t)}{dt} + bv(t) = 0 \]  

(6)

The initial condition is given by equation (3) above, repeated here for convenience:

\[ v(t=0) = v_0 \]

Equation (6) is a homogeneous differential equation, since there is no forcing function applied to the system. Thus, the particular solution in this case is \( y_p(t) = 0 \) and our overall solution is simply the homogeneous solution, \( y(t) = y_h(t) \).

To solve the above differential equation, we rearrange equation (6) to give:

\[ \frac{m}{b} \frac{dv(t)}{dt} = -v(t) \]  

(7)

Separating variables in equation (7) results in:

\[ \frac{dv(t)}{v} = -\frac{b}{m} dt \]  

(8)

Incorporating dummy variables of integration and integrating both sides of (8) gives:
\[
\int_{v_0}^{v(t)} \frac{d\xi(t)}{\xi} = -\frac{b}{m} \int_0^t d\xi
\]

which evaluates to

\[
\ln(\frac{v(t)}{v_0}) = -\frac{b}{m} t \Rightarrow \ln[v(t)] - \ln[v_0] = -\frac{b}{m} t \Rightarrow \ln\left[\frac{v(t)}{v_0}\right] = -\frac{b}{m} t
\]

Taking the exponent of both sides of the above provides our final result:

\[
v(t) = v_0 e^{-bt/m}
\]  

(9)

A plot of the response given in equation (9) is shown in Figure 3. This plot matches our previous expectations, the velocity of the mass at time \( t = 0 \) is \( v_0 \) and the velocity decreases exponentially until the mass is (essentially) at rest. Referring to chapter 2.1.1, we see that the response of equation (10) can be written in terms of a time constant as:

\[
v(t) = v_0 e^{-t/\tau}
\]

Where the time constant \( \tau = \frac{m}{b} \). This result also agrees with our intuition: as the friction coefficient decreases, the time constant increases and the mass comes to rest more slowly. Likewise, increasing the mass causes the time constant to increase – a larger mass will tend to "coast" for a longer time.

![Figure 3. Homogeneous response of mass-damper system.](image)
Case ii: Response to step input

We will now consider the case in which the mass is initially at rest, and a constant force is applied to the mass at time $t = 0$. Intuitively, we expect the velocity of the mass to increase to some final value; the final velocity of the mass corresponds to the condition in which the frictional force is equal and opposite to the applied mass (recall that in our model, the frictional force is proportional to velocity – as the velocity increases, the frictional force opposing the motion also increases). We now solve the governing differential equation for this system and compare the results to our expectations.

The differential equation governing the system, valid for $t > 0$, and initial condition, providing the energy in the system at $t = 0$, are:

$$m \frac{dv(t)}{dt} + bv(t) = F$$

$$v(t = 0) = 0$$

where $F$ is the magnitude of the (constant) applied force. Note that since $F$ is constant, and only applied for times $t > 0$, we have a step input with magnitude $F$. We want to solve the above differential equation for $t > 0$; since the input forcing function can be represented as a step function, this resulting solution is called the step response of the system.

For this case, we have both a nonzero forcing function and an initial condition to consider. Thus, we must determine both the homogeneous and particular solutions and superimpose the result per equation (3) above.

The homogeneous solution is determined from:

$$m \frac{dv_h(t)}{dt} + bv_h(t) = 0$$

where $v_h(t)$ is the homogeneous solution. This equation has been solved as case i; the form of the solution is

$$v_h(t) = K_1 e^{-bt/m}$$

Note:

The velocity of the mass tells us how much kinetic energy is being stored by the system. The initial condition provides the energy initially stored in the system. The calculated response describes how this energy is dissipated through the sliding friction. No energy is added to the system in this case, since the external applied force is zero.
where \( K_1 \) is (in this case) an unknown constant which will be determined from our initial conditions.

The particular solution is determined from:

\[
m \frac{dv_p(t)}{dt} + bv_p(t) = F
\]

where \( v_p(t) \) is the particular solution to the differential equation in equation (10). Since the right-hand side of equation (13) is constant for \( t > 0 \), the left-hand side of the equation must also be constant for \( t > 0 \) and \( v_p(t) \) must be constant for \( t > 0 \). If \( v_p(t) \) is constant, \( \frac{dv_p(t)}{dt} \) is zero and equation (13) simplifies to:

\[
bv_p(t) = F
\]

so that

\[
v_p(t) = \frac{F}{b}
\]

Superimposing equations (12) and (14), per the principle expressed in equation (3) results in:

\[
v(t) = v_h(t) + v_p(t) = K_1 e^{-bt/m} + \frac{F}{b}
\]

We can now use our initial condition, \( v(t=0) = 0 \), to determine the constant \( K_1 \). Evaluating equation (15) at \( t = 0 \) and applying the initial condition results in:

\[
v(t=0) = 0 = K_1 e^{-b0/m} + \frac{F}{b}
\]

Since \( e^{-b0/m} = 1 \), equation (16) results in:

\[
K_1 = -\frac{F}{b}
\]

Substituting equation (17) into equation (5) results in the overall solution

\[
v(t) = -\frac{F}{b} e^{-bt/m} + \frac{F}{b} = \frac{F}{b} \left( 1 - e^{-bt/m} \right)
\]

If, as in case i, we define the time constant \( \tau = \frac{m}{b} \), equation (13) can be expressed as:
A plot of the system response is shown in Figure 4. This plot matches our intuitive expectations: the initial velocity is zero; the applied force causes the mass to move. When the frictional and applied forces balance, the velocity of the mass becomes constant. The time constant is determined by the mass and the frictional coefficient; a larger mass results in a longer time constant – it takes longer to get a large mass to its final velocity than a small mass. The frictional coefficient also affects the system time constant; a smaller friction coefficient results in a longer time constant. This result seems counter-intuitive at first, since a smaller frictional coefficient should allow us to accelerate the mass more rapidly. However, the smaller frictional coefficient also results in a higher final velocity – since the time constant is defined by the time required to reach approximately 63.2% of the final velocity, the higher final velocity causes a longer time constant even though the mass is accelerating more rapidly. (If the damping coefficient is zero, the time constant goes to infinity. However, the final velocity also goes to infinity – it takes an infinite amount of time to get to 63.2% of an infinite velocity!)

Figure 4. Step response of mass-damper system.

Note:

The velocity of the mass again describes the energy stored by the system; in this case, the initial velocity is zero and the system has no energy before the force is applied. The applied force adds energy to the system by causing the mass to move. When the rate of energy addition by the applied force and energy dissipation by the friction balance, the velocity of the mass becomes constant and the energy stored in the system becomes constant.
Summary:

We use the results of the above examples to re-state some primary results in more general terms. It is seen above that the natural and step responses of first order systems are strongly influenced by the system time constant, \( \tau \). The original, general, differential equation – equation (1) above – can be re-written directly in terms of the system time constant. We do this by dividing equation (1) by the coefficient \( a_i \). This results in:

\[
\frac{dy(t)}{dt} + \frac{a_0}{a_1} y(t) = \frac{1}{a_1} f(t), \quad t > t_0
\]  

(20)

Defining \( \tau = \frac{a_i}{a_0} \), equation (20) becomes:

\[
\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) = \frac{1}{a_1} f(t)
\]  

(21)

The initial condition on equation (21) is as before:

\[ y(t = t_0) = y_0 \]  

(22)

The cases of the system homogenous response (or natural or unforced response) and step response are now stated more generally, for the system described by equations (21) and (22).

i. **Homogeneous response**

For the homogenous response \( f(t) = 0 \), and the system response is

\[ y(t) = y_0 e^{-\frac{t}{\tau}}, \text{ for } t \geq 0. \]  

(23)

The response is shown graphically in Figure 5.
2.4.1: Introduction to First Order System Responses

![First-order system homogeneous response](image)

Figure 5. First-order system homogeneous response.

ii. **Step response**

For a step input of amplitude $A$, $f(t) = Au_0(t)$ where $u_0(t)$ is the unit step function defined in chapter 2.1. Substituting this input function into equation (21),

$$\frac{dy(t)}{dt} + \frac{1}{\tau} y(t) = \frac{A}{a_1}, \quad t > 0$$  \hspace{1cm} (24)

Using the approach of case ii of our previous mass-damper system example, we determine the system response to be:

$$y(t) = \frac{A}{a_0} \left[ 1 - e^{-\tau / \tau} \right]$$  \hspace{1cm} (25)

This response is shown graphically in Figure 6.

![First order system response to step input with amplitude A](image)

Figure 6. First order system response to step input with amplitude $A$. 
Key points:

- A first order system is described by a first order differential equation. The order of the differential equation describing a system is the same as the number of independent energy storage elements in the system – a first order system has one independent energy storage element. (The number of energy dissipation elements is arbitrary, however.)

- The differential equation governing a first order system is of the form:
  \[ a_1 \frac{dy(t)}{dt} + a_0 y(t) = f(t) \]
  where \( y(t) \) is the system output, \( f(t) \) is the applied input to the system, and \( a_0 \) and \( a_1 \) are constants.

- The differential equation governing a first order system can also be written in the form
  \[ \frac{dy(t)}{dt} + \frac{1}{\tau} y(t) = \frac{f(t)}{a_1} \]
  where \( y(t) \) is the system output, \( f(t) \) is the applied input to the system, and \( \tau \) is the system time constant.

- The system time constant is a primary parameter used to describe the response of first order systems.

- In this chapter, we considered two types of forcing functions: a zero-input case, in which \( f(t) = 0 \) and \( y(t=0) = y_0 \), and a step input case, in which \( f(t) = Au_0(t) \) and \( y(t=0) = 0 \). For the zero-input case, the response is:
  \[ y(t) = y_0 e^{-\frac{t}{\tau}}, \quad t>0 \]
  For the step input case, the so-called step response is:
  \[ y(t) = \frac{A}{a_0} \left[ 1 - e^{-\frac{t}{\tau}} \right], \quad t>0 \]

- The system response consists of two parts: a homogeneous solution and a particular solution. The response can also be considered to consist of a transient response and a steady-state response. The homogeneous solution and the transient response die out with time; they are due to a combination of the system characteristics and the initial conditions. The particular solution and the steady state response have the same form as the forcing function; they persist as \( t \to \infty \). It can be seen from the above that, for the zero-input case, the steady state response is zero (since the forcing function is zero). The steady state step response is \( \frac{A}{a_0} \); it is a constant value and is proportional to the magnitude of the input forcing function.