2.5.1: Second Order Circuits

Overview

Second order systems are, by definition, systems whose input-output relationship is a second order differential equation. A second order differential equation contains a second order derivative but no derivative higher than second order. Second order systems contain two independent energy storage elements, per our previous comments pertaining to the relationship between the number of energy storage elements in a system and the system order.

Second order systems, like first order systems, are an extremely important class of systems. In previous chapters, we saw that the natural response of first order systems decays exponentially with time – the natural response decays monotonically to zero. The natural response of second order systems can, however, oscillate with time – we will see that a second order systems response can contain sinusoidal components. The motion of a pendulum, for example, can be modeled by a second order system. These oscillations are due to the transfer of energy between the two energy storage mechanisms; a pendulum, for example, oscillates because of the cyclic exchange of potential and kinetic energy of the mass.

Before beginning this chapter, you should be able to:

- Write the voltage-current relations for capacitors in integral and differential form from memory (Chapter 2.2)
- Write voltage-current relations for inductors in integral and differential form from memory (Chapter 2.3)
- Write equations governing first order electrical circuit natural response (Chapters 2.4.2, 2.4.3)

After completing this chapter, you should be able to:

- Write differential equations governing second order parallel RLC circuits
- Write differential equations governing second order series RLC circuits
- Define damping ratio and natural frequency from the coefficients of a second order differential equation

This chapter requires:

- N/A
We will develop our discussion of second order systems in the context of two electrical circuits examples.

**Example 1: Series RLC circuit:**

Consider the circuit shown in Figure 1 below, consisting of a resistor, a capacitor, and an inductor (this type of circuit is commonly called an RLC circuit). The circuit contains two energy storage elements: and inductor and a capacitor. The energy storage elements are independent, since there is no way to combine them to form a single equivalent energy storage element. Thus, we expect the governing equation for the circuit to be a second order differential equation. We will develop equations governing both the capacitor voltage, $v_C(t)$ and the inductor current, $i_L(t)$ as indicated in Figure 1.

![Figure 1. Series RLC circuit](image)

In order to determine the governing equations for $v_C(t)$ and $i_L(t)$ we will attempt to write two first-order differential equations for the system and then combine these equations to obtain the desired second order differential equation. To facilitate this process, the circuit of Figure 1 is repeated in Figure 2 with the node and loop we will use labeled. Note that we also label the current through the capacitor in terms of the capacitor voltage and the voltage across the inductor in terms of the inductor current.

![Figure 2. Series RLC circuit with node and loop defined.](image)
The voltage-current relationships for inductors and capacitors indicate that, in Figure 2,
\[ i_c(t) = C \frac{dv_c(t)}{dt} \quad \text{and} \quad v_i(t) = L \frac{di_i(t)}{dt}. \]
Using the latter of these relations, KVL around the indicated loop in Figure 2 provides:
\[ v_i(t) = R i_L(t) + v_c(t) + L \frac{di_L}{dt}. \]  
(1)

KCL at node A, along with the voltage-current relation for the capacitor, indicates that:
\[ C \frac{dv_c(t)}{dt} = i_L(t). \]  
(2)

Important Tip:
Equations (1) and (2) consist of two coupled first order differential equations in two unknowns: \( i_i(t) \) and \( v_c(t) \). This set of differential equations completely describes the behavior of the circuit — if we are given appropriate initial conditions and the input function \( v_s(t) \) they can be solved to determine the inductor currents and capacitor voltages. Once the capacitor voltage and inductor current are known, the energy in the system is completely defined and we can determine any other desired circuit parameters. Any manipulations of equations (1) and (2) we perform subsequently in this chapter do not fundamentally increase the information we have about the circuit — we will simply be trying to rearrange equations (1) and (2) to make it easier (in some ways) to interpret the circuit behavior.

We can determine the equation governing the capacitor voltage by differentiating equation (2) with respect to time to obtain an expression for the derivative of the inductor current:
\[ C \frac{d^2v_c(t)}{dt^2} = \frac{di_L(t)}{dt}. \]  
(3)

Substituting equations (2) and (3) into equation (1) results in:
\[ v_s(t) = RC \frac{dv_c(t)}{dt} + v_c(t) + LC \frac{d^2v_c(t)}{dt^2}. \]
Rearranging this slightly results in
\[ \frac{d^2v_c(t)}{dt^2} + \frac{R}{L} \frac{dv_c(t)}{dt} + \frac{1}{LC} v_c(t) = \frac{1}{LC} v_s(t). \]  
(4)

To determine the relationship governing the inductor current, we can again use equation (2) to write the capacitor voltage as:
\[ v_C(t) = \frac{1}{C} \int_{0}^{t} i_L(t) \, dt \]  

where we assume that the voltage across the capacitor at time \( t = 0 \) is zero; e.g. \( v_C(0) = 0 \).

Substituting equation (5) into equation (1) results in the integro-differential equation:

\[ v_i(t) = R i_L(t) + \frac{1}{C} \int_{0}^{t} i_L(t) \, dt + L \frac{d i_L}{d t} \]

In general, we prefer not to work with a mixture of derivatives and integrals in the same equation, so we differentiate the above to obtain our final expression for \( i_L(t) \):

\[ \frac{d^2 i_L(t)}{d t^2} + \frac{R}{L} \frac{d i_L(t)}{d t} + \frac{1}{LC} i_L(t) = \frac{1}{L} \frac{d v_S(t)}{d t} \]

\[ (6) \]

**Example 2: Parallel RLC circuit**

Our second exemplary circuit is the parallel combination of a resistor, capacitor, and inductor shown in Figure 3. The circuit is, for relatively obvious reasons, called a parallel RLC circuit. The forcing function to the circuit is provided by a current source, \( i_S(t) \). The circuit of Figure 3, like that of Figure 2, contains two independent energy storage elements—we expect the governing equations for the circuit to be second order differential equations. We will again develop equations governing both the capacitor voltage, \( v_C(t) \) and the inductor current, \( i_L(t) \) as indicated in Figure 3.

![Parallel RLC circuit](image)

**Figure 3. Parallel RLC circuit.**

Consistent with our approach for the series RLC circuit, we will write first order differential equations using the variables \( v_C(t) \) and \( i_L(t) \) and subsequently combine these equations to eliminate the undesired unknown. Figure 4 shows the node and loop we will use to generate these equations. Figure 4 also shows the current through the capacitor in terms of the capacitor voltage and the voltage across the inductor in terms of the inductor current.
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Figure 4. Parallel RLC circuit with node and loop defined.

KVL around the indicated loop provides:

\[ L \frac{di_L(t)}{dt} = v_C(t) \]  \hspace{1cm} (7)

KCL at node A provides:

\[ i_s(t) = \frac{v_C(t)}{R} + i_L(t) + C \frac{dv_C(t)}{dt} \]  \hspace{1cm} (8)

As in example 1, equations (7) and (8) completely describe the circuit’s response. However, to gain additional insight into the individual parameters \( v_C(t) \) and \( i_L(t) \), we rearrange these equations into second order differential equations in a single dependent variable. For example, we can differentiate equation (7) to obtain

\[ L \frac{d^2i_L(t)}{dt^2} = \frac{dv_C(t)}{dt} \]  \hspace{1cm} (9)

Equations (7) and (9) can be substituted into equation (8) to obtain a second order differential equation in the variable \( i_L(t) \). After some manipulation, the resulting equation is

\[ \frac{d^2i_L(t)}{dt^2} + \frac{1}{RC} \frac{di_L(t)}{dt} + \frac{1}{LC} i_L(t) = \frac{1}{LC} i_s(t) \]  \hspace{1cm} (10)

Likewise, we can integrate equation (7) and use the result to write equation (8) in terms of the capacitor voltage:

\[ \frac{d^2v_C(t)}{dt^2} + \frac{1}{RC} \frac{dv_C(t)}{dt} + \frac{1}{LC} v_C(t) = \frac{1}{C} \frac{dv_S(t)}{dt} \]  \hspace{1cm} (11)

The important thing to note about the above results is that equations (4), (6), (10), and (11) can all be written in the form:
\[
\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = f(t)
\] 

(12)

where \(y(t)\) is any system parameter of interest (for example, a voltage or current in an electrical circuit), \(\omega_n\) is the undamped natural frequency and \(\zeta\) is the damping ratio; the physical significance of these parameters will be presented later in this series of chapters. The point being made here is that the governing equation for any second order system can be written in the form of equation (12); thus, we will focus on this format for our discussion of the solution of second order differential equations.