Overview

Sinusoidal signals and complex exponentials are extremely important to any engineer who is concerned with determining the dynamic response of a system. Electrical circuits, in particular, are often characterized by their response to sinusoidal inputs.

This chapter provides some background relative to these signals.

Before beginning this chapter, you should be able to:

- Write expressions for sinusoidal functions
- Express complex numbers in rectangular and polar form

After completing this chapter, you should be able to:

- Write complex numbers in terms of complex exponentials
- Express sinusoidal signals in terms of complex exponentials

This chapter requires:

- N/A
Sinusoidal Signals:

Sinusoidal signals are represented in terms of sine and/or cosine functions. In general, we will represent sinusoids as cosine functions. Our general expression for a sinusoidal signal is:

\[ v(t) = V_p \cos(\omega t + \theta) \]  

(1)

where \( V_p \) is the zero-to-peak amplitude of the sinusoid, \( \omega \) is the radian frequency of the sinusoid (we will always use radians/second as the units of \( \omega \)) and \( \theta \) is the phase angle of the sinusoid (in units of either radians or degrees are used for phase angle – recall that \( 2\pi \) radians = \( 360^\circ \)). A representative plot of a sinusoidal signal is provided in Figure 1. In Figure 1, the frequency of the sinusoid is indicated as a period of the signal (the period is defined as the shortest time interval at which the signal repeats itself). The radian frequency of a sinusoid is related to the period by:

\[ \omega = \frac{2\pi}{T} \]  

(2)

Figure 1. Arbitrary sinusoidal signal.

Note:

Complex exponential signals have both real and imaginary parts; when we introduce complex exponentials later in this chapter, we will see that the cosine function is the real part of a complex exponential signal. Complex exponentials make dynamic systems analysis relatively simple – thus, we often analyze a signals response in terms of complex exponentials. Since any measurable quantity is real-valued, taking the real part of the analytical result based on complex exponentials will result in a cosine function. Thus, cosines become a natural way to express signals which vary sinusoidally.
The frequency of a sinusoidal signal is alternately expressed in units of Hertz (abbreviated Hz). A Hertz is the number of cycles which the sinusoid goes through in one second. Thus, Hertz correspond to cycles/second. The frequency of a signal in Hertz is related to the period of the signal by

\[ f = \frac{1}{T} \]  

(3)

Radian frequencies relate to frequencies in Hertz by:

\[ f = \frac{2\pi}{\omega} \implies \omega = 2\pi f \]  

(4)

Although frequencies of signals are often expressed in Hertz, it is not a unit which lends itself to calculations. Thus, all our calculations will be performed in radian frequency – if given a frequency in Hertz, it should be converted to radians/second before any calculations are performed based on this frequency.

**Complex Exponentials:**

In our presentation of complex exponentials, we first provide a brief review of complex numbers. A complex number contains both real and imaginary parts. Thus, we may write a complex number \( A \) as:

\[ A = a + jb \]  

(5)

where

\[ j = \sqrt{-1} \]  

(6)

The complex number \( A \) can be represented on orthogonal axes representing the real and imaginary part of the number, as shown in Figure 2. (In Figure 2, we have taken the liberty of representing \( A \) as a vector, although it is really just a number.) We can also represent the complex number in polar coordinates, also shown in Figure 2. The polar coordinates consist of a magnitude \(|A|\) and phase angle \( \theta_A \), defined as:

\[ |A| = \sqrt{a^2 + b^2} \]  

(7)

\[ \theta_A = \tan^{-1}\left(\frac{b}{a}\right) \]  

(8)

Notice that the phase angle is defined counterclockwise from the positive real axis. Conversely, we can determine the rectangular coordinates from the polar coordinates from

\[ a = Re\{A\} = |A| \cos(\theta_A) \]  

(9)
where the notation $Re\{A\}$ and $Im\{A\}$ denote the real part of $A$ and the imaginary part of $A$, respectively.

The polar coordinates of a complex number $A$ are often represented in the form:

$$A = |A| \angle \theta_A$$  \hspace{1cm} (11)

Figure 2. Representation of a complex number in rectangular and polar coordinates.

An alternate method of representing complex numbers in polar coordinates employs complex exponential notation. Without proof, we claim that

$$e^{j\theta} = 1 \angle \theta$$  \hspace{1cm} (12)

Thus, $e^{j\theta}$ is a complex number with magnitude 1 and phase angle $\theta$. From Figure 2, it is easy to see that this definition of the complex exponential agrees with Euler’s equation:

$$e^{\pm j\theta} = \cos \theta \pm j \sin \theta$$  \hspace{1cm} (13)

With the definition of equation (12), we can define any arbitrary complex number in terms of complex numbers. For example, our previous complex number $A$ can be represented as:

$$A = |A|e^{j\theta_A}$$  \hspace{1cm} (14)

We can generalize our definition of the complex exponential to time-varying signals. If we define a time varying signal $e^{j\omega t}$, we can use equation (13) to write:

$$e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t$$  \hspace{1cm} (15)
The signal $e^{j\omega t}$ can be visualized as a unit vector rotating around the origin in the complex plane; the tip of the vector scribes a unit circle with its center at the origin of the complex plane. This is illustrated in Figure 3. The vector rotates at a rate defined by the quantity $\omega$ -- the vector makes one complete revolution every $\frac{2\pi}{\omega}$ seconds. The projection of this rotating vector on the real axis traces out the signal $\cos \omega t$, as shown in Figure 3, while the projection of the rotating vector on the imaginary axis traces out the signal $\sin \omega t$, also shown in Figure 3.

Thus, we interpret the complex exponential function $e^{j\omega t}$ as an alternate “type” of sinusoidal signal. The real part of this function is $\cos \omega t$ while the imaginary part of this function is $\sin \omega t$.