Overview

So far, we have written differential equations governing a system as a single differential equation in a single unknown. These differential equations are called input-output equations, since they describe the output of the system in terms of an input or forcing function. This approach is useful for low-order systems, such as the first or second order systems we have examined so far, but it becomes cumbersome for higher-order systems.

*State variable or state space* models of dynamic systems have some advantages over input-output system models. In the state variable model approach, we represent an $N^{th}$ order system as $N$, first order differential equations rather than one $N^{th}$ order differential equation. This tends to simplify the modeling of higher order systems. Numerical simulation of state variable models is also considerably simpler than the numerical simulation of input-output equations; the differential equation solvers in MATLAB, for example, require the equations to be in state variable form. State variable models have a number of other advantages over input-output models, none of which are of immediate interest to us.

Before beginning this chapter, you should be able to:

- Write differential equations governing first order electrical circuits (Chapters 2.4.1 – 2.4.5)
- Write differential equations governing second order electrical circuits (Chapters 2.5.1 – 2.5.5)

After completing this chapter, you should be able to:

- Define state variables for electrical circuits
- Write differential equations governing electrical circuits in state variable form

This chapter requires:

- N/A
Background and Introduction:

As their name implies, state variable models are based on the concept of a system’s state. The *state* of a system is the minimum amount of information necessary to completely characterize the system at some instant in time. It turns out that the system’s state provides the information necessary to uniquely determine the energy in all the system’s energy storage elements and vice-versa. If the energy in any of the energy storage elements changes, the system’s state changes.

The *state variables* are the smallest set of variables which completely describe the state (or the energy storage) of the system. The choice of state variables is not unique, but one possible choice of state variable is those variables which describe the energy stored in all of the independent energy storage elements in the system. For example, in electrical circuits, inductors store energy as current and capacitors store energy as voltage. If we choose as state variables the currents in inductors and the voltages across the capacitors, we will have created a legitimate set of state variables for the circuit.

Since the state variables are independent, they can be visualized as a set of orthogonal axes defining a space. The space defined by the state variables is called *state space* of the system. If the system is described by $N$ state variables, the state space will be $N$-dimensional. The state of the system at any given time can be visualized as a point in the state space.

In general, as we apply an input to a system, the system’s state will change over time. Since the state of the system is a point in state space, the change in the system’s state can be visualized as tracing a path over time in the state space. This path is called the *state trajectory*.

Form of State Variable Models:

State variable models, as mentioned previously, represent an $N^{th}$ order system as $N$ first order differential equations in $N$ unknowns. (The unknowns are the state variables.) For linear, time invariant systems, these equations will take the form:

$$
\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1N}x_N + b_1u \\
\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2N}x_N + b_2u \\
\vdots \\
\dot{x}_n = a_{N1}x_1 + a_{N2}x_2 + \cdots + a_{NN}x_N + b_Nu
$$

where $x_1, x_2, \cdots, x_N$ are the system states and $u$ is the input to the system. The overdot notation denotes differentiation with respect to time; $\dot{x}_k = \frac{dx_k}{dt}$. (We will assume that no derivatives of the input are applied to the system – this is a special case which we will avoid in this introductory chapter.)

It turns out that, if all system states are known, we can determine any other parameter in the system. In fact, any other parameter in the system can be written as a linear combination of the states and the input. Thus, we can write the system output as:
Equations (2) and (3) are commonly written in matrix form as:

\[
\begin{align*}
\dot{x}(t) &= Ax(t) + bu(t) \\
y(t) &= c^T x(t) + du(t)
\end{align*}
\]

(3) 
(4)

In equation (3), the vector \(x(t)\) is an \(N\times1\) column vector containing the system state variables. The matrix \(A\) is a square \(N\times N\) matrix, and the vector \(b\) is an \(N\times1\) column vector. The vector \(\dot{x}(t)\) is an \(N\times1\) column vector containing the derivatives of the state variables as a function of time. In equation (4), the vector \(c\) is a \(1\times N\) row vector, and \(d\) is a scalar. Equation (3) provides the state equations for the system, and equation (4) is called the output equation of the system.

Creation of the state variable model for an electrical circuit is probably best described by an example.

Example: State Variable Model for Electric Circuit:

Consider the circuit shown below. The input to the system is the voltage \(u(t)\), and the output variable is the voltage across the resistor, \(y(t)\). There are three energy storage elements in the system (two inductors and a capacitor) so we will expect the system to be third order with three state variables. These state variables are chosen to be the currents through the inductors and the voltage difference across the capacitor; these are indicated on the figure below.

We write the state equations by applying KVL and KCL to the circuit. Since there are three state variables, three state equations must be written. Applying KVL around the leftmost loop results in:

\[u(t) = L_1 \dot{x}_1 + x_3\]

Applying KVL around the rightmost loop results in:

\[x_3 = L_2 \dot{x}_2 + Rx_2\]

Note that in the equation above, the voltage across the resistor is written as \(Rx_2\), rather than \(y\). This is consistent with the general state equation format which requires that the derivative of each state variable be written only in terms of the other state variables and the input. Our final state equation is obtained by applying KCL at the node interconnecting the two inductors and the capacitor:
\[ x_1 = x_2 + C\dot{x}_3 \]

The above can be re-written in matrix form as:

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{bmatrix} =
\begin{bmatrix}
0 & 0 & -\frac{1}{L_1} \\
0 & -\frac{R}{L_2} & \frac{1}{L_2} \\
\frac{1}{C} & -\frac{1}{C} & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} +
\begin{bmatrix}
\frac{1}{L_1} \\
0 \\
0
\end{bmatrix} u(t)
\]

The above state equations allow us to determine any parameter of interest in the circuit. Our output variable is the voltage across the resistor, \( R \). We can use Ohm’s law to write the equation describing the desired output in terms of the state variable \( x_2 \) to obtain:

\[
y(t) = \begin{bmatrix}
0 & R & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3
\end{bmatrix} + 0 \cdot u(t)
\]