Overview

In this module, we will review properties of sinusoidal functions and complex exponentials. We will also introduce phasor notation, which will significantly simplify the sinusoidal steady-state analysis of systems, and provide terminology which will be used in subsequent sinusoidal steady-state related modules.

Most of the material presented here has been provided previously in chapter 2.5.3; this material will, however, be important enough to bear repetition. Likewise, a brief overview of complex arithmetic, which will be essential in using complex exponentials effectively, is provided at the end of this module. Students who need to review complex arithmetic may find it useful to peruse this overview before reading the section of this module relating to complex exponentials and phasors.

Before beginning this module, you should be able to:

• Write the equation governing an arbitrary cosine function
• Sketch the sinusoid corresponding to a given cosine function
• Perform complex arithmetic

After completing this module, you should be able to:

• Define periodic signals
• Define the amplitude, frequency, radian frequency, and phase of a sinusoidal signal
• Express sinusoidal signals in phasor form

This module requires:

• N/A

Sinusoidal Signals:

The sinusoidal signal shown in Figure 1 is represented mathematically by:

\[ f(t) = V_p \cos(\omega t) \]  (1)

The amplitude or peak value of the function is \(V_p\). \(V_p\) is the maximum value achieved by the function; the function itself is bounded by \(+V_p\) and \(-V_p\), so that \(-V_p \leq f(t) \leq V_p\). The radian frequency or angular frequency of the function is \(\omega\); the units of \(\omega\) are radians/second. The function is said to be periodic; periodic functions repeat themselves at regular intervals, so that

\[ f(t + nT) = f(t) \]  (2)

where \(n\) is any integer and \(T\) is the period of the signal. The sinusoidal waveform shown in Figure 1 goes through one complete cycle or period in \(T\) seconds. Since the sinusoid of equation (1) repeats itself every \(2\pi\) radians, the period is related to the radian frequency of the sinusoid by:
2.7.1: Sinusoidal signals, complex exponentials, and phasors

\[ \omega = \frac{2\pi}{T} \]  

(3)

It is common to define the frequency of the sinusoid in terms of the number of cycles of the waveform which occur in one second. In these terms, the frequency \( f \) of the function is:

\[ f = \frac{1}{T} \]  

(4)

The units of \( f \) are cycles/second or hertz (abbreviated Hz). The frequency and radian frequency are related by

\[ f = \frac{\omega}{2\pi} \]  

(5)

or equivalently,

\[ \omega = 2\pi f \]  

(6)

Regardless of whether the sinusoid’s rate of oscillation is expressed as frequency or radian frequency, it is important to realize that the argument of the sinusoid in equation (1) must be expressed in radians. Thus, equation (1) can be expressed in terms of frequency in Hz as:

\[ f(t) = \cos(2\pi ft) \]  

(7)

To avoid confusion in our mathematics, we will almost invariably write sinusoidal functions in terms of radian frequency as shown in equation (1), although Hz is generally taken as the standard unit for frequency (experimental apparatus, for example, commonly express frequency in Hz).

![Figure 1. Cosine waveform.](image-url)
A more general expression of a sinusoidal signal is

\[ v(t) = V_p \cos(\omega t + \theta) \]  

(8)

where \( \theta \) is the phase angle or phase of the sinusoid. The phase angle simply translates the sinusoid along the time axis, as shown in Figure 2. A positive phase angle shifts the signal left in time, while a negative phase angle shifts the signal right – this is consistent with our discussion of step functions in chapter 2.1, where it was noted that subtracting a value from the unit step argument resulted a time delay of the function. Thus, as shown in Figure 2, a positive phase angle causes the sinusoid to be shifted left by \( \frac{\theta}{\omega} \) seconds.

The units of phase angle should be radians, to be consistent with the units of \( \omega t \) in the argument of the cosine. It is typical, however, to express phase angle in degrees, with 180° corresponding to \( \pi \) radians. Thus, the conversion between radians and degrees can be expressed as:

\[
\text{Number of degrees} = \frac{180}{\pi} \times \text{Number of radians}
\]

For example, we will consider the two expressions below to be equivalent, though the expression on the right-hand side of the equal sign contains a mathematical inconsistency:

\[ V_p \cos(\alpha + \frac{\pi}{2}) = V_p \cos(\alpha + 90^\circ) \]

Figure 2. Cosine waveform with non-zero phase angle.
For convenience, we introduce the terms leading and lagging when referring to the sign on the phase angle, $\theta$. A sinusoidal signal $v_1(t)$ is said to lead another sinusoid $v_2(t)$ of the same frequency if the phase difference between the two is such that $v_1(t)$ is shifted left in time relative to $v_2(t)$. Likewise, $v_1(t)$ is said to lag another sinusoid $v_2(t)$ of the same frequency if the phase difference between the two is such that $v_1(t)$ is shifted right in time relative to $v_2(t)$. This terminology is described graphically in Figure 3.

Finally, we note that the representation of sinusoidal signals as a phase shifted cosine function, as provided by equation (8), is completely general. If we are given a sinusoidal function in terms of a sine function, it can be readily converted to the form of equation (8) by subtracting a phase of $\frac{\pi}{2}$ (or 90°) from the argument, since:

$$\sin(\omega t) = \cos(\omega t - \frac{\pi}{2})$$

Likewise, sign changes can be accounted for by a $\pm\pi$ radian phase shift, since:

$$-\cos(\omega t) = \cos(\omega t \pm \pi)$$

Obviously, we could have chosen either a cosine or sine representation of a sinusoidal signal. We prefer the cosine representation, since a cosine is the real part of a complex exponential. In the next module, we will see that sinusoidal steady-state circuit analysis is simplified significantly by using complex exponentials to represent the sinusoidal functions. The cosine is the real part of a complex exponential (as we saw previously in chapter 2.5.3). Since all measurable signals are real valued, we take the real part of our complex exponential-based result as our physical response; this results in a solution of the form of equation (8).

Since representation of sinusoidal waveforms as complex exponentials will become important to us in circuit analysis, we devote the following subsection to a review of complex exponentials and their interpretation as sinusoidal signals.
Complex Exponentials and Phasors:

Euler’s identity can be used to represent complex numbers as complex exponentials:

\[ e^{\pm j\theta} = \cos \theta \pm j \sin \theta \]  

(9)

If we generalize equation (9) to time-varying signals of arbitrary magnitude we can write:

\[ V_p e^{\pm j(\omega t + \theta)} = V_p \cos(\omega t + \theta) \pm jV_p \sin(\omega t + \theta) \]  

(10)

so that

\[ V_p \cos(\omega t + \theta) = \text{Re}\{V_p e^{\pm j(\omega t + \theta)}\} \]  

(11)

and

\[ V_p \sin(\omega t + \theta) = \text{Im}\{V_p e^{\pm j(\omega t + \theta)}\} \]  

(12)

where \( \text{Re}\{V_p e^{\pm j(\omega t + \theta)}\} \) and \( \text{Im}\{V_p e^{\pm j(\omega t + \theta)}\} \) denote the real part of \( V_p e^{\pm j(\omega t + \theta)} \) and the imaginary part of \( V_p e^{\pm j(\omega t + \theta)} \), respectively.

The complex exponential of equation (10) can also be written as:

\[ V_p e^{j(\omega t + \theta)} = V_p e^{j\theta} e^{j\omega t} \]  

(13)

The term \( V_p e^{j\theta} \) on the right-hand side of equation (13) is simply a complex number which provides the magnitude and phase information of the complex exponential of equation (10). From equation (11), this magnitude and phase can be used to express the magnitude and phase angle of a sinusoidal signal of the form given in equation (8).

The complex number in polar coordinates which provides the magnitude and phase angle of a time-varying complex exponential, as given in equation (13) is called a phasor. The phasor representing \( V_p \cos(\omega t + \theta) \) is defined as:

\[ \underline{V} = V_p e^{j\theta} = V_p e^{j\theta} \]  

(14)

We will use a capital letter with an underscore to denote a phasor. Using bold typeface to represent phasors is more common; our notation is simply for consistency between lecture material and written material – boldface type is difficult to create on a whiteboard during lecture!

Note:

The phasor representing a sinusoid does not provide information about the frequency of the sinusoid – frequency information must be kept track of separately.
Complex Arithmetic Review:

The bulk of the material in this section is taken from chapter 2.5.3. It is repeated here for convenience.

In our presentation of complex exponentials, we first provide a brief review of complex numbers. A complex number contains both real and imaginary parts. Thus, we may write a complex number \( A \) as:

\[
A = a + jb
\]

where

\[
j = \sqrt{-1}
\]

and the underscore denotes a complex number. The complex number \( A \) can be represented on orthogonal axes representing the real and imaginary part of the number, as shown in Figure 4. (In Figure 4, we have taken the liberty of representing \( A \) as a vector, although it is really just a number.) We can also represent the complex number in polar coordinates, also shown in Figure 4. The polar coordinates consist of a magnitude \( |A| \) and phase angle \( \theta_A \), defined as:

\[
|A| = \sqrt{a^2 + b^2}
\]

\[
\theta_A = \tan^{-1}\left(\frac{b}{a}\right)
\]

Notice that the phase angle is defined counterclockwise from the positive real axis. Conversely, we can determine the rectangular coordinates from the polar coordinates from

\[
a = Re\{A\} = |A| \cos(\theta_A)
\]

\[
b = Im\{A\} = |A| \sin(\theta_A)
\]

where the notation \( Re\{A\} \) and \( Im\{A\} \) denote the real part of \( A \) and the imaginary part of \( A \), respectively.

The polar coordinates of a complex number \( A \) are often represented in the form:

\[
A = |A|\angle\theta_A
\]
2.7.1: Sinusoidal signals, complex exponentials, and phasors

An alternate method of representing complex numbers in polar coordinates employs complex exponential notation. Without proof, we claim that

$$e^{j\theta} = 1\angle \theta$$

(22)

Thus, $e^{j\theta}$ is a complex number with magnitude 1 and phase angle $\theta$. From Figure 4, it is easy to see that this definition of the complex exponential agrees with Euler’s equation:

$$e^{\pm j\theta} = \cos \theta \pm j \sin \theta$$

(23)

With the definition of equation (22), we can define any arbitrary complex number in terms of complex numbers. For example, our previous complex number $A$ can be represented as:

$$A = |A|e^{j\theta}$$

(24)

We can generalize our definition of the complex exponential to time-varying signals. If we define a time varying signal $e^{j\omega t}$, we can use equation (23) to write:

$$e^{\pm j\omega t} = \cos \omega t \pm j \sin \omega t$$

(25)

The signal $e^{j\omega t}$ can be visualized as a unit vector rotating around the origin in the complex plane; the tip of the vector scribes a unit circle with its center at the origin of the complex plane. This is illustrated in Figure 3. The vector rotates at a rate defined by the quantity $\omega$ -- the vector makes one complete revolution every $\frac{2\pi}{\omega}$ seconds. The projection of this rotating vector on the real axis traces out the signal $\cos \omega t$, as shown in Figure 3, while the projection of the rotating vector on the imaginary axis traces out the signal $\sin \omega t$, also shown in Figure 5.

Thus, we interpret the complex exponential function $e^{j\omega t}$ as an alternate “type” of sinusoidal signal. The real part of this function is $\cos \omega t$ while the imaginary part of this function is $\sin \omega t$. 

Figure 4. Representation of a complex number in rectangular and polar coordinates.

$\Im$ $\Re$

$$|A|\sin(\theta_A)$$ $|A|\cos(\theta_A)$

$$b$$

$$a$$
Addition and subtraction of complex numbers is most easily performed in rectangular coordinates. Given two complex numbers \( A \) and \( B \), defined as:

\[
A = a + jb \\
B = c + jd
\]

the sum and difference of the complex number can be determined by:

\[
A + B = ( a + c ) + j(b + d)
\]

and

\[
A - B = ( a - c ) + j(b - d)
\]
2.7.1: Sinusoidal signals, complex exponentials, and phasors

Multiplication and division, on the other hand, are probably most easily performed using polar coordinates. If we define two complex numbers as:

\[ A = |A|e^{j\theta_A} = |A|\angle \theta_A \]
\[ B = |B|e^{j\theta_B} = |B|\angle \theta_B \]

the product and difference can be determined by:

\[ A \cdot B = |A|e^{j\theta_A} \cdot |B|e^{j\theta_B} = |A| \cdot |B|e^{j(\theta_A + \theta_B)} = |A| \cdot |B|\angle (\theta_A + \theta_B) \]

and

\[ \frac{A}{B} = \frac{|A|e^{j\theta_A}}{|B|e^{j\theta_B}} = \frac{A}{B}e^{j(\theta_A - \theta_B)} = \frac{A}{B} \angle (\theta_A - \theta_B) \]

The conjugate of a complex number, denoted by a *, is obtained by changing the sign on the imaginary part of the number. For example, if \( A = a + jb = |A|e^{j\theta} \), then

\[ A^* = a - jb = |A|e^{-j\theta} \]

Conjugation does not affect the magnitude of the complex number, but it changes the sign on the phase angle. It is easy to show that

\[ A \cdot A^* = |A|^2 \]

Several useful relationships between polar and rectangular coordinate representations of complex numbers are provided below. The student is encouraged to prove any that are not self-evident.

\[ j = 1\angle 90^\circ \]
\[- j = 1\angle -90^\circ \]
\[ \frac{1}{j} = -j = 1\angle -90^\circ \]
\[ 1 = 1\angle 0^\circ \]
\[ -1 = 1\angle 180^\circ \]