Overview

It is common to categorize circuits by the overall shape of their magnitude response. As we saw in example 8 of chapter 2.8.1, in some frequency ranges the output amplitude may be high relative to the input amplitude, while in other frequency ranges the output amplitude will be low relative to the input amplitude. If the output amplitude at some frequency is high relative to the input amplitude, the magnitude response at that frequency is “large” and that frequency is said to be passed by the circuit. Conversely, if the output amplitude at some frequency is low relative to the input amplitude, the magnitude response at that frequency is “small” and that frequency is said to be stopped by the circuit.

Circuits which select certain frequencies to pass and other frequencies to stop are called frequency selective circuits or filters (since they tend to “filter out” certain frequency ranges of the input signal). The range of frequencies which are passed are called the passband of the filter, and the range of frequencies which are stopped are called the stopband of the filter. There are four primary categories of filters:

- **Low-pass filters** pass low frequencies and stop high frequencies
- **High-pass filters** pass high frequencies and stop low frequencies
- **Band-pass filters** pass a range of frequencies between two ranges of stopped frequencies
- **Band-reject filters** stop a range of frequencies between two ranges of passed frequencies

Filters are also categorized by their order. The order of the filter is simply the order of the differential equation governing the filter. Thus first-order filters are governed by a first order differential equation, a second-order filter is governed by a second-order differential equation, and so on. Low-pass and high-pass filters can be any order, while band-pass and band-stop filters must be at least second order.

In this chapter, we restrict our attention to first order filters, so we will consider only low pass and high pass filters.

Before beginning this module, you should be able to:

- Calculate the frequency response of an electrical circuit (Chapter 2.8.0)
- Calculate and plot the magnitude and phase responses of a circuit (Chapter 2.8.0, 2.8.1)
- Plot the spectrum of a signal (Chapter 2.8.1)

After completing this module, you should be able to:

- Identify low pass and high pass filters
- Calculate a system’s cutoff frequency
- Determine the DC gain of an electrical circuit

This module requires:

- N/A
Ideal low-pass and high-pass filters:

We will first introduce the basic concepts relative to first order filters in the context of ideal filters. It must be clearly understood that ideal filters are not physically realizable – that is, we cannot construct a physical system which can perform this way\(^1\). Ideal filters entirely stop all input signals in the stopband and completely pass all signals in the passband. Thus, the magnitude response of an ideal filter is exactly one in the passband and exactly zero in the stop band. First order filters can be either high-pass or low-pass filters.

An ideal low pass filter has a magnitude response as shown in Figure 1. The passband is shown as the shaded area under the magnitude response. The magnitude response is discontinuous – it goes from one to zero instantaneously. The cutoff frequency, \(\omega_c\), defines the boundary between the passband and the stopband. Any signal with a frequency below \(\omega_c\) is passed through the filter without any attenuation; any signal with a frequency above \(\omega_c\) is entirely stopped by the filter – it is not present in the output signal.

![Figure 1. Magnitude response of an ideal low-pass filter.](image)

An ideal high-pass filter has a magnitude response as shown in Figure 2. The passband is again shown as the shaded area under the magnitude response. The magnitude response is discontinuous – it goes from zero to one instantaneously. The cutoff frequency, \(\omega_c\), again defines the boundary between the passband and the stopband. Any signal with a frequency below \(\omega_c\) is entirely stopped by the filter while any signal with a frequency above \(\omega_c\) is passed through the filter with no amplitude change.

As previously noted, it is impossible to physically implement an ideal filter. Thus, all electrical circuits implement non-ideal filters. Non-ideal filters do not provide an instantaneous transition between the pass band and the stop band. Non-ideal first order filters are discussed in the following subsections.

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\(^1\) In order to implement an ideal filter, the circuit would need to know values of the input signal before the values are applied – that is, the circuit would need to be able to see into the future. The technical term is that the filter would need to be non-causal. It is, of course, impossible to build a device that predicts the future.
First order low pass filters:

The form of the governing differential equation for a first order low pass filter can be written as:

\[
\frac{dy(t)}{dt} + \omega_c y(t) = K \cdot u(t)
\]  \hspace{1cm} (1)

The frequency response of the filter can be determined to be

\[
H(\ j\omega) = \frac{K}{j\omega + \omega_c}
\]  \hspace{1cm} (2)

The magnitude response of the filter is thus

\[
|H(\ j\omega)| = \frac{K}{\sqrt{\omega^2 + \omega_c^2}}
\]  \hspace{1cm} (3)

The maximum magnitude response is \(\frac{K}{\omega_c}\) when \(\omega=0\) and the magnitude response is zero as \(\omega \to \infty\). Thus, the filter is passing low frequencies and stopping high frequencies. The differential equation describes a low-pass filter.

The magnitude response of the filter is shown in Figure 3. The frequency response is a smooth curve, rather than the discontinuous function shown in Figure 1. There is no single frequency that obviously separates the passband from the stopband, so we must choose a relatively arbitrary point to define the boundary between the passband and the stopband. By consensus, the cutoff frequency for a low-pass filter is defined as the frequency at which the magnitude response is \(\frac{1}{\sqrt{2}}\) times the magnitude response at \(\omega=0\). For the magnitude response given by equation (3), the cutoff frequency is \(\omega_c\). This point is indicated on Figure 3.
First order high pass filters:

The form of the governing differential equation for a first order low pass filter can be written as:

$$\frac{dy(t)}{dt} + \omega_c y(t) = K \cdot \frac{du(t)}{dt}$$

(4)

The frequency response of the filter can be determined to be

$$H(j\omega) = \frac{jK\omega}{j\omega + \omega_c}$$

(5)

The magnitude response of the filter is thus

$$|H(j\omega)| = \frac{K\omega}{\sqrt{\omega^2 + \omega_c^2}}$$

(6)

The maximum magnitude response is approximately $K$ when $\omega \rightarrow \infty$ and the magnitude response is zero at $\omega = 0$. Thus, the filter is passing high frequencies and stopping low frequencies. The differential equation describes a high-pass filter.

The magnitude response of the filter is shown in Figure 4. As with the non-ideal low-pass filter, the frequency response is a smooth curve, rather than the discontinuous function shown in Figure 2. Again, there is no single frequency that obviously separates the passband from the stopband, so we must choose a relatively arbitrary point to define the boundary between the passband and the stopband. Consistent with our choice of cutoff frequency for the low-pass filter, the cutoff frequency for a high-pass filter is defined as the frequency at which the magnitude response is $\frac{1}{\sqrt{2}}$ times the
magnitude response at $\omega \rightarrow \infty$. For the magnitude response given by equation (6), the cutoff frequency is $\omega = \omega_c$. This point is indicated on Figure 4.

![Graph](image)

Figure 3. Magnitude response of first order high-pass filter.

**Notes:**

- The cutoff frequency is also called the corner frequency, the 3dB frequency, or the half-power point.

- The cutoff frequency for both low-pass and high-pass filters is defined as the frequency at which the magnitude is $\frac{1}{\sqrt{2}}$ times the maximum value of the magnitude response.

- It can be seen from chapter 2.8.1 that the phase response of a first order low-pass filter is $0^\circ$ at $\omega = 0$ and decreases to $-90^\circ$ as $\omega \rightarrow \infty$. The phase response is $-45^\circ$ at the cutoff frequency.

- It can be seen from chapter 2.8.1 that the phase response of a first order high-pass filter is $90^\circ$ at $\omega = 0$ and decreases to $0^\circ$ as $\omega \rightarrow \infty$. The phase response is $45^\circ$ at the cutoff frequency.

- For both low-pass and high-pass filters, the cutoff frequency is the inverse of the time constant for the circuit, so that $\omega_c = \frac{1}{\tau}$.

- The circuit’s response at zero frequency is generally an important parameter to consider. This is called the DC gain, and is the ratio of the output amplitude to the input amplitude for a constant input. A constant input corresponds to a cosine with zero frequency. Low pass filters have a relatively high DC gain and a correspondingly large response to a constant input. High pass filters have a low DC gain; they have little or no response to constant inputs.

We conclude this section with examples of circuits from chapter 2.8.1 which implement low-pass and high-pass filter operations.
Example 1: First order low-pass filter

The circuit below is the circuit from example 3 of chapter 2.8.1. The input is $v_{in}(t)$ and the output is $v_{out}(t)$.

![Circuit Diagram]

In chapter 2.8.1, the frequency response was determined to be:

$$H(j\omega) = \frac{2}{2 + j\omega}$$

The maximum value of the magnitude response is one at a frequency of zero radians/second and the magnitude response goes to infinity as $\omega \to \infty$, so the circuit acts as a low-pass filter.

Comparing the amplitude response above with equation (2) above, it can be seen that the cutoff frequency is $\omega_c = 2$ rad/sec. The amplitude response, with the cutoff frequency labeled, is shown below.
Example 2: First order high-pass filter

The circuit below is the circuit from example 4 of chapter 2.8.1. The input is $v_s(t)$ and the output is $v(t)$.

The frequency response of the circuit was previously determined to be:

$$H(j\omega) = \frac{V}{V_s} = \frac{j2\omega}{1 + j2\omega}$$

The maximum value of the magnitude response is one as $\omega \to \infty$ and goes to zero at a frequency of zero radians/second, so the circuit acts as a low-pass filter. Comparing the amplitude response above with equation (4) above, it can be seen that the cutoff frequency is $\omega_c = 0.5 \text{ rad/sec}$. The amplitude response, with the cutoff frequency labeled, is shown below.